Spacetime Dependent Lagrangians and the Barriola-Vilenkin Monopole Mass

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Abstract

We apply the spacetime dependent lagrangian formalism [1] to the action in general relativity. We obtain Barriola-Vilenkin [B-V] [2] type of topological solution by exploiting the electro-gravity duality [4] of the vacuum Einstein equations. The monopole mass M is shown to be of order $a/\mathtt{G}$ with $a/2\mathtt{G} < M < a/\mathtt{G}$, a a small positive constant and $\mathtt{G}$ Newton's gravitational constant. The lower of these bounds can be written as $M \geq (a/2\mathtt{G})\beta$, where $(1/2) < \beta(= 1+2k) < 1$, and $2k$ is the global monopole charge.This reminds us of the Bogomolny bound for usual monopoles. Comparing with the Barriola-Vilenkin scenario of a spontaneously broken scalar field theory ,this can also provide an experimental check on the Grand Unification Scale (presently $10^{16} Gev$) if the B-V monopole is ever discovered.

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1. Introduction

Recently it has been shown that electromagnetic duality as well as weak-strong (field-theoretic) duality in equations of motion can be obtained by introducing explicit spacetime dependence of the lagrangian [1]. Duality symmetry (which hinges on nonperturbative aspects of a theory) is implemented through certain general set of characteristics of a spacetime dependent function (representing the spacetime dependence of the lagrangian) at large values of the spacetime coordinates (i.e. on the boundary). The solutions for the fields obtained under these conditions are topological in nature.Electrogravity duality [4] is a topological defect generating process. Under eletrogravity duality transformations on the Einstein equations, vacuum solutions get mapped onto solutions with a global monopole. So it is natural to ask whether the formalism in [1] can accommodate this. We show that our formalism [1], after incorporating the electrogravity duality invariance of the vacuum equations, leads to the famous Barriola-Vilenkin solution [2].The monopole mass M is shown to be bounded as $a/2\mathtt{G} < M < a/\mathtt{G}$. a is a small positive constant and $\mathtt{G}$ is Newton's gravitational constant. Various aspects of topolological defects can be found in [3].

In the literature [4] electrogravity duality is related to the fact that the vacuum Einstein equations $G_{ik} = 0$ imply $R_{ik} = 0$ where $G_{ik} = R_{ik} - (1/2)g_{ik}R$ is the Einstein tensor, R_{ik} is Ricci tensor,R is the Ricci scalar and G the Einstein scalar. So the vacuum Einstein equations are invariant under $G_{ik} \leftrightarrow R_{ik}$. According to Dadhich *et al* this means that (like the electromagnetic field) the gravitational field can be resolved into "electric" (due to charge) and "magnetic" (motion of charge) parts [4]. For gravity the analogues are mass-energy and its motion respectively. Gravity entails two kinds of charges– non-gravitational matter-energy (”active” part) and gravitational field energy (”passive” part).Electrogravity duality means that vacuum equations are invariant under the interchange of the active and passive ”electric” parts. Technically this means interchange of Ricci and Einstein curvatures.Dadhich [4] have done exhaustive investigations and also shown that the B-V solution can be obtained from this.

In our formalism we take the (vacuum) equations of motion for electrogravity duality as

$$R_{ik} - (1/2)g_{ik}R = 0 \tag{1a}$$

$$G_{ik} - (1/2)g_{ik}G = 0 \tag{1b}$$

(Note:$G_{ik} = R_{ik} - (1/2)g_{ik}R$ and $R_{ik} = G_{ik} - (1/2)g_{ik}G$) and stipulate that that either or both of the above equations are always satisfied in the presence of non-gravitational matter-energy sources with the action modified to the form shown in equation (4) (below) and field equations given by equation (2) (below).[($1a$) is the analogue of $\partial_i F^{ik} = 0$ and ($1b$) that of $\partial_i \tilde{F}^{ik} = 0$ in Ref.1a]. Usually , ($1a$) is obtained via the variational principle from S and then assuming that the energy momentum tensor $T_{ik} = 0$. Note that whereas $\tilde{F}^{ik}$ is the Hodge dual of F^{ik}, the same is not true for G_{ik} with respect to R_{ik}. The duality transformation maps the Ricci tensor into the Einstein tensor and vice versa. This is because contraction of the Riemann tensor is the Ricci while its double dual is the Einstein tensor [4].

We first recall our formalism [1]. Let the lagrangian L' be a function of fields η_ρ, their derivatives $\eta_{\rho,\nu}$ *and the spacetime coordinates* x_ν, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$. Variational principle yields : $\int dV \left(\partial_\eta L' - \partial_\mu \partial_{\partial_\mu \eta} L'\right) = 0$. Assuming a separation of variables : $L'(\eta_\sigma, \eta_{\sigma,\nu}, ..x_\nu) = \Lambda(x_\nu) L(\eta_\sigma, \eta_{\sigma,\nu})$.

($\Lambda(x_\nu)$ is the x_ν dependent part and is a finite non-vanishing function) gives

$$\int dV \left(\partial_\eta(\Lambda L) - \partial_\mu \partial_{\partial_\mu \eta}(\Lambda L)\right) = 0 \tag{2}$$

The usual action in gravity is (in units where velocity of light is 1)

$$S = -(1/16\pi \mathtt{G}) \int d^4x R\sqrt{-g} + (1/2) \int d^4x T_{ik}\sqrt{-g} \tag{3}$$

g is the determinant of the metric tensor, $\mathtt{G}$ is Newton's gravitational constant. ($1a$) follows from the variational principle when $T_{ik} = 0$. We now show that the modified action

$$S_\Lambda = -(1/16\pi \mathtt{G}) \int d^4x \Lambda(x) R\sqrt{-g} \tag{4}$$

leads to the Barriola-Vilenkin monopole metric for large values of r once we impose the conditions that (i)Λ is determined such that ($1a$) is obeyed, (ii) Λ is finite at infinity (i.e.boundary) and (iii) the equations of motion are given by (2). This is how the Barriola-Vilenkin solutions are accommodated in our formalism. Note that there is no T_{ik} term in the modified action S_Λ. Electrogravity duality as embodied in (1) will be sufficient to generate the solution. The spacetime dependence (after the overall dependence on R) is expressed by Λ (assumed to be a function of r only; $r \to \infty$ is a boundary of the theory). Λ is *not* dynamical and is a finite, non-vanishing function at all x_ν. It is like some external field and equations of motion for Λ meaningless at the length scales (classical gravity) under consideration.We show that Λ is finite at infinity. The finite behaviour of Λ on the boundary *encodes the electrogravity duality of the theory within the boundary.* In this way we are reminded of the holographic principle [5]. From (2) and (4) the equations of motion that follow are

$$-(1/16\pi \mathtt{G})[\Lambda(R_{ik} - (1/2)g_{ik}R) - (1/\sqrt{-g})M_{ik}] = 0 \tag{5a}$$

$$M_{ik} = (1/2)\sqrt{-g}(\partial_m\Lambda)g^{nm}(\partial_n g_{ik}) - \partial_n(g_{ik}g^{nm}\sqrt{-g}\ \partial_m\Lambda)$$

$$+\partial_q(g_i^m g_k^q\ \partial_m\Lambda\ \sqrt{-g}) - (1/2)\sqrt{-g}\ \partial_i\Lambda\ g^{pq}(\partial_q g_{kp} + \partial_p g_{kq} - \partial_k g_{pq}) \quad (5b)$$

where we have used $\Gamma^i_{jk} = (1/2)g^{im}[\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}]$ as the affine connections. The electrogravity duality equations $(1a)$ are obtained if $M_{ik} = 0$. Assume that $M_{ik} = 0$ for all i, k except $i = k = 0$. We now find a Λ for which $M_{00} = 0$. Take the spherically symmetric metric outside some mass distribution M.

$$ds^2 = g_{00}dt^2 - g_{rr}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

$$\equiv B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (6)$$

where $A(r)B(r) = 1$. Time components are "00" and radial components "rr". r is the Schwarzschild radial coordinate where the area of the surface of the sphere, $r = constant$, is given by $4\pi r^2$. Then

$$M_{00} = (1/2)\sqrt{-g}\ \partial_r\Lambda\ g^{rr}\ \partial_r g_{00} - \partial_r(g^{rr}g_{00}\sqrt{-g}\ \partial_r\Lambda) \quad (7)$$

Using $AB = 1$, $\sqrt{-g} = r^2 sin\theta$, $M_{00} = 0$ then implies

$$\frac{d^2\Lambda}{dr^2} + \frac{d\Lambda}{dr}\ \Big(\frac{2}{r}\ +\ \frac{3}{2}\ \frac{1}{B}\ \frac{dB}{dr}\Big) = 0 \quad (8)$$

$$g_{00} = B(r) = A(r)^{-1} = g_{rr}^{-1} = (b/r^2)^{2/3}(\frac{d\Lambda}{dr})^{-2/3} = e^{-P} \quad (9)$$

b is a constant. By inspection a solution for Λ which satisfies the relevant conditions is

$$\Lambda(r) = \frac{b}{a_0[a_1 + a_2/r]^{1/2}}$$

where $B(r) = a_1 + a_2/r$. We now show that the above solution may be identified with that corresponding to the usual spherically symmetric metric outside some mass distribution M via the Einstein equations. We introduce e^{-P} as per conventions. The Einstein equation corresponding to g_{00} is then

$$\frac{d}{dr}(re^{-P}) - 1 = 0 \tag{10}$$

where $P = -lng_{00} = -lnB(r)$. Now g_{00} must be such that Λ is finite at ∞. (10) integrates to

$$e^{-P} = B(r) = g_{00} = g_{rr}^{-1} = A(r)^{-1} = 1 + constant/r$$

The constant is determined following usual methods to be $-2\mathtt{G}M$. Hence

$$\Lambda(r) = \frac{-b}{\mathtt{G}M}\Big(1 - \frac{2\mathtt{G}M}{r}\Big)^{-1/2} = \frac{a}{\mathtt{G}M}\Big(1 - \frac{2\mathtt{G}M}{r}\Big)^{-1/2} \tag{11}$$

where $a = -b$ is taken to be a positive constant and we take $\Lambda(r = 0) = 0$. This solution satisfies the conditions (i)-(iii) mentioned before. With this solution consider the integrand of (4) and write: (n is some positive integer)

$$\Lambda(x)\sqrt{-g} = \sqrt{-\Lambda^2 g}$$

$$= \sqrt{-\Lambda^{-1/n} g_{00}\ \ \Lambda^{1/n} g_{11}\ \ \Lambda g_{22}\ \ \Lambda g_{33}} = \sqrt{-g^*_{00} g^*_{11} g^*_{22} g^*_{33}} = \sqrt{-g^*}$$

If the action is to be compared with the usual GTR action, then R should be determined with the new g^*_{ik}. We first find g^*_{ik} and then R^*_{ik} and R^*. (*Note: we are not considering conformal transformations of the metric.*) Then

$$g^*_{00} = \Lambda^{-1/n} g_{00} = (\mathtt{G}M/a)^{1/n}[1 - 2\mathtt{G}M/r]^{(2n+1)/2n}$$

$$= 1 - 2\mathtt{G}M/r + [(\mathtt{G}M/a)^{1/n} - 1] \tag{12}$$

where we have neglected $O(2\mathtt{G}M/r)^2$ terms and chosen the constant a such that $(\mathtt{G}M/a)^{1/n} = 2n/(2n+1) = 1/[1+1/(2n)] \ < \ 1$. The limit $lim_{n\to\infty}[1+$

$1/(2n)]^n$ exists finitely and lies between 1 and 2. This means that $a/(\mathtt{G}M)$ can be made of *order unity* by choosing n sufficiently large. Therefore

$$a/2\mathtt{G} < M < a/\mathtt{G} \tag{13}$$

This is the principal result of this work. So $M \approx a/G$. The lower bound may be written as $M \geq (a/2\mathtt{G})\beta$, where $(1/2) < \beta (= 1 + 2k) < 1$, $2k$ is the global monopole charge (see below after equation (18)). This reminds us of the usual Bogomolny bound $M_{monopole} \geq vg$ (v is the vacuum expectation value of the Higgs and g the monopole charge) for usual monopoles. The monopole mass can be made very small as a is a free parameter.This result is consistent with that of Harari and Lousto [3b], while avoiding the physically undesirable feature of monopole mass becoming negative. Further, if we choose a to be small then for large r we have :

$$g^*_{00} = 1 - 2\mathtt{G}M/r - \alpha \tag{14a}$$

$$g^*_{11} = [1 - 2\mathtt{G}M/r - \alpha]^{-1} \tag{14b}$$

$$g^*_{22} = [a/(\mathtt{G}M)]r^2 + ar + a \quad \approx \quad r^2 \tag{14c}$$

$$g^*_{33} \approx r^2 sin^2\theta \tag{14d}$$

where $\alpha = [1 - (\mathtt{G}M/a)^{1/n}]$ is now positive. So for large r and large n

$$ds^2 = (1 - 2\mathtt{G}M/r - \alpha)dt^2 - (1 - 2\mathtt{G}M/r - \alpha)^{-1}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{15a}$$

This is the Barriola-Vilenkin metric for a global monopole [2]. The action (4) gets modified into

$$S^* = -(1/16\pi\mathtt{G}) \int d^4 x R^* \sqrt{-g^*} \tag{15b}$$

If we take the factor $a/(\mathtt{G}M)$ in (14c) to be of order unity, neglect the factor $2\mathtt{G}M/r$ and do the rescaling of coordinates as in [2], then a measure of the deficit solid angle [2] is given by $2n/(2n+1)$.The surface $\theta = \pi/2$ has the geometry of a cone with a deficit angle $2n\pi/(2n+1)$. All the observational consequences discussed elegantly in Ref.2 can now be parametrised in terms of the integer n. Another point is that comparing with the scenario of Ref.[2], $\alpha \equiv 8\pi \mathtt{G}\eta^2$, where η is the Grand Unification Scale of order $10^{16} Gev$. Then $\eta^2 \approx (1/8\pi\mathtt{G})[1-(\mathtt{G}M/a)^{1/n}]$. So if the monopole is ever discovered and M determined ,an independent check on η will be obtained. The Ricci and Einstein tensor components are

$$R^*_{00} = R^*_{11} = 0 \ ; \ R^*_{22} = -\alpha; \ R^*_{33} = -\alpha sin^2\theta; \tag{16a}$$

$$R^{0*}_0 = R^{1*}_1 = R^*_{01} = 0 \ ; R^{2*}_2 = R^{3*}_3 = -\alpha/r^2 \ ; \ R^* = -2\alpha/r^2 \tag{16b}$$

$$G^*_{00} = G^*_{11} = (1 - 2\mathtt{G}M/r - \alpha)(\alpha/r^2) \ ; \ G^*_{22} = G^*_{33} = 0; \tag{17a}$$

$$G^{0*}_0 = G^{1*}_1 = \alpha/r^2 \ ; G^{2*}_2 = G^{3*}_3 = G^*_{01} = 0 \ ; \ G^* = 2\alpha/r^2 \tag{17b}$$

Electrogravity duality means interchange of the Ricci and Einstein curvatures.The most interesting vacuum solutions are the Schwarzschild solutions and they are also unique. In obtaining these solutions, it turns out that one does not need to use all the equations; i.e.there remains one equation free which is implied by the others [4]. For the Schwarzschild solution the vacuum equation ultimately reduces to the two equations (ϕ is Newtonian potential) $-\nabla^2_r\phi = 0$ and $-(2/r^2)\frac{\partial}{\partial r}(r\phi) = 0$. The latter equation implies the former. Hence, even if we modify the vacuum equations by putting something on the right hand side, the equations will still yield Schwarzschild as the unique solution and the modified equation can as well characterize vacuum for spherical symmetry. *However, the modified equation would now no longer be duality* *invariant* i.e. $R^*_{ik} - (1/2)g^*_{ik}R^* \neq 0$. The solution of the dual set (i.e. dual-vacuum) is the Schwarzschild vacuum with global monopole charge. We now show this w.r.t. the equations (16).

$$R^{0*}_0 = -\nabla^2_r\phi = 0 \ \ and \ \ R^{2*}_2 = -(2/r^2)\frac{\partial}{\partial r}(r\phi) = -\alpha/r^2 \tag{18}$$

The first of the equations in (18) integrates to give $\phi(r) = -k_1/r - k = -2\mathtt{G}M/r - k(say)$ where k_1 and k are constants of integration. Substituting this in the second equation will give $2k/r^2 = -\alpha/r^2$ i.e.$\alpha = -2k$. So we have identified α with the global monopole charge. For $\mathtt{G}M/a$ of order unity (n large), $\alpha = 1 - (\mathtt{G}M/a)^{1/n} \approx 1 - \mathtt{G}M/a$. So (13) implies $1/2 < (1 + 2k) < 1$. Thus, the vacuum equations are no more duality symmetric and the dual vacuum is the Schwarzschild vacuum with global monopole charge.

Next consider the set (17). Duality transformation implies that R^*_{ik} is replaced by G^*_{ik}. So

$$G^{0*}_0 = -\nabla^2_r\phi = \alpha/r^2 \ \ and \ \ G^{2*}_2 = -(2/r^2)\frac{\partial}{\partial r}(r\phi) = 0 \tag{19}$$

The second of the equations in (19) integrates to give $\phi(r) = -k_1/r = -2\mathtt{G}M/r(say)$ where k_1 is a constant of integration. Substituting this in the first equation will give $0 = -\alpha/r^2$ i.e.$\alpha = 0$. It is easily seen that $G^*_{ik} - (1/2)g^*_{ik}G^* = 0$ are satisfied by this solution. So we have got back the usual Schwarzschild solution, *with the vacuum equations given by* (1*b*). Our stipulation that either of the equations in (1) should hold true is thus satisfied. (Both are true only when $R = -G = 0$). So at large r (i.e. boundary), the action which represents the Schwarzschild solution is (note $G = -R$)

$$S^* = -(1/16\pi\tilde{\mathtt{G}})\int d^4x(-G^*)\sqrt{-g^*} \tag{20}$$

where the equation of motion is now $(1b)$ and $\tilde{\mathtt{G}} = -\mathtt{G}$ (also suggested by Dadhich [4])-so that the action may be cast in the usual form. Hence the theory on the boundary represents an action in terms of the Einstein tensor whose solution gives the usual Schwarzschild solution while the dual theory (in terms of the Ricci tensor) within the boundary represents a Schwarszchild black hole with a global monopole charge. So for $r \to \infty, \Lambda \to a/(\mathtt{G}M) \approx 1$, R is replaced by $-G$ and $\mathtt{G}$ by $-\mathtt{G}$ and we are again reminded of the holographic principle as described in the case of electromagnetic and weak-strong duality [1a,b]. The possibility of constructing noncommuting coordinates [5] on the boundary will depend on whether the above theory has a quantum version.

So the formalism in [1] can also accommodate electrogravity duality. The B-V monopole mass is predicted to be $\approx a/G$ with a a free parameter. An analogue of the Bogomolny bound (present in usual monopoles) is provided for the B-V monopole.Further, the possibility of an independent experimental check on the Grand Unification scale exists.

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